Language: English

7^{th} Middle European Mathematical Olympiad

Individual Competition

24^{th} August 2013

Problem I-1. Let a, b, c be positive real numbers such that

$$a+b+c=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

Prove that

$$2(a+b+c)\geq \sqrt[3]{7a^2b+1}+\sqrt[3]{7b^2c+1}+\sqrt[3]{7c^2a+1}.$$

Find all triples (a,b,c) for which equality holds.

Problem I-2. Let n be a positive integer. On a board consisting of $4n\times 4n$ squares, exactly $4n$ tokens are placed so that each row and each column contains one token. In a step, a token is moved horizontally or vertically to a neighbouring square. Several tokens may occupy the same square at the same time. The tokens are to be moved to occupy all the squares of one of the two diagonals.

Determine the smallest number $k(n)$ such that for any initial situation, we can do it in at most $k(n)$ steps.

Problem I-3. Let ABC be an isosceles triangle with $AC=BC$. Let N be a point inside the triangle such that $2\angle ANB=180^\circ+\angle ACB$. Let D be the intersection of the line BN and the line parallel to AN that passes through C. Let P be the intersection of the angle bisectors of the angles CAN and ABN.

Show that the lines DP and AN are perpendicular.

Problem I-4. Let a and b be positive integers. Prove that there exist positive integers x and y such that

$$\binom{x+y}{2}=ax+by.$$

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.