Language: English

# 7th Middle European Mathematical Olympiad

Team Competition

25th August 2013

**Problem T-1.** Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x) + 2y) = f(x^2) + f(y) + x + y - 1$$

for all $x$, $y \in \mathbb{R}$.

**Problem T-2.** Let $x$, $y$, $z$, $w \in \mathbb{R} \setminus \{0\}$ such that $x + y \neq 0$, $z + w \neq 0$, and $xy + zw \geq 0$. Prove the inequality

$$\left(\frac{x+y}{z+w} + \frac{z+w}{x+y}\right)^{-1} + \frac{1}{2} \geq \left(\frac{x}{z} + \frac{z}{x}\right)^{-1} + \left(\frac{y}{w} + \frac{w}{y}\right)^{-1}.$$

**Problem T-3.** There are $n \geq 2$ houses on the northern side of a street. Going from the west to the east, the houses are numbered from 1 to $n$. The number of each house is shown on a plate. One day the inhabitants of the street make fun of the postman by shuffling their number plates in the following way: for each pair of neighbouring houses, the current number plates are swapped exactly once during the day.

How many different sequences of number plates are possible at the end of the day?

**Problem T-4.** Consider finitely many points in the plane with no three points on a line. All these points can be coloured red or green such that any triangle with vertices of the same colour contains at least one point of the other colour in its interior.

What is the maximal possible number of points with this property?

**Problem T-5.** Let $ABC$ be an acute triangle. Construct a triangle $PQR$ such that $AB = 2PQ$, $BC = 2QR$, $CA = 2RP$, and the lines $PQ$, $QR$, and $RP$ pass through the points $A$, $B$, and $C$, respectively. (All six points $A$, $B$, $C$, $P$, $Q$, and $R$ are distinct.)

**Problem T-6.** Let $K$ be a point inside an acute triangle $ABC$, such that $BC$ is a common tangent of the circumcircles of $AKB$ and $AKC$. Let $D$ be the intersection of the lines $CK$ and $AB$, and let $E$ be the intersection of the lines $BK$ and $AC$. Let $F$ be the intersection of the line $BC$ and the perpendicular bisector of the segment $DE$. The circumcircle of $ABC$ and the circle $k$ with centre $F$ and radius $FD$ intersect at points $P$ and $Q$.

Prove that the segment $PQ$ is a diameter of $k$.

**Problem T-7.** The numbers from 1 to $2013^2$ are written row by row into a table consisting of $2013 \times 2013$ cells. Afterwards, all columns and all rows containing at least one of the perfect squares $1, 4, 9, \ldots, 2013^2$ are simultaneously deleted.

How many cells remain?

**Problem T-8.** The expression

$$\pm \square \pm \square \pm \square \pm \square \pm \square \pm \square$$

is written on the blackboard. Two players, $A$ and $B$, play a game, taking turns. Player $A$ takes the first turn. In each turn, the player on turn replaces a symbol $\square$ by a positive integer. After all the symbols $\square$ are replaced, player $A$ replaces each of the signs $\pm$ by either $+$ or $-$, independently of each other. Player $A$ wins if the value of the expression on the blackboard is not divisible by any of the numbers $11, 12, \ldots, 18$. Otherwise, player $B$ wins.

Determine which player has a winning strategy.

*Time: 5 hours*
*Time for questions: 60 min*
*Each problem is worth 8 points.*
*The order of the problems does not depend on their difficulty.*